Symmetric tensors with applications to Hilbert schemes

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Abstract. Let $A[X]_U$ be a fraction ring of the polynomial ring $A[X]$ in the variable X over a commutative ring A. We show that the Hilbert functor $\mathcal{H}ilb^n_{A[X]_U/A}$ is represented by an affine scheme $\mathrm{Symm}^n_A(A[X]_U)$ given as the ring of symmetric tensors of $\otimes^n_A A[X]_U$. The universal family is given as $\mathrm{Symm}^{n-1}_A(A[X]_U) \times_A \mathrm{Spec}(A[X]_U)$.

§1. - Introduction.

The Hilbert functor $\mathcal{H}ilb^n_{\mathcal{O}_{V,x}}$ parameterizing closed subschemes of a variety V, having finite length n and support in a fixed point $x \in V$, has been studied by several authors in the last decades ([2], [3], [4], [6], [7], [8] and [13]). The main interest have been on the classification of the set of k-rational points of $\mathcal{H}ilb^n_{\mathcal{O}_{V,x}}$. The scheme structure on the parameter schemes was apparently neglected until [13], where the Hilbert functor $\mathcal{H}ilb^n_{k[X]_{(X)}}$ parameterizing subschemes of the line of finite length n and support in the origin, was described.

It was shown in [13] that the scheme representing the functor $\mathcal{H}ilb^n_{k[X]_{(X)}}$ is of dimension n. Taking into account that the set of k-rational points of $\mathcal{H}ilb^n_{k[X]_{(X)}}$ is trivial, the scheme structure on the parameter scheme was a surprise. The Hilbert scheme parameterizing subschemes of the line, having finite length and support in one point is not an algebraic scheme, that is a scheme which is not of finite type over the base.

The motivation behind the present paper arose from the desire to better understand the techniques introduced in [13]. Instead of only consider finite length subschemes of the line with support in a fixed point, we allow the subschemes to have support in any given set.

Let $A[X]_U$ be the fraction ring of the polynomial ring $A[X]$ in the variable X over a base ring A, with respect to a multiplicatively closed subset $U \subseteq A[X]$. We study the contravariant functor $\mathcal{H}ilb^n_{A[X]_U/A}$ from the category of A-schemes to sets, which sends an A-scheme T to the set of closed subschemes $Z \subseteq T \times_A \mathrm{Spec}(A[X]_U)$ such that the projection map $p : Z \to T$ is flat and where the global sections of the fiber $p^{-1}(t)$ is of rank n for all points $t \in T$.

Our main result is Theorem (8.2), where we show that $\mathcal{H}ilb^n_{A[X]_U/A}$ is represented by the n-fold symmetric product $\mathrm{Symm}^n_A(A[X]_U)$ of $\mathrm{Spec}(A[X]_U)$. Thus we obtain

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a result which is similar to what is known for Hilbert schemes of points on smooth projective curves [5], and similar with the results of B. Iversen about the n-fold sections for smooth families of curves [9].

When studying the functor $\mathcal{H}ilb^n_{A[X]_U/A}$ the key problem is to determine those monic polynomials $F(X)$ in $A[X]$ such that the fraction map $A[X]/(F(X)) \to A[X]_U/(F(X))$ is an isomorphism. A problem which is solved by the use of the symmetric operators [13] of the polynomial ring $A[X]$ associated to $F(X)$. We define, Section (2.2), an A-algebra homomorphism u_F from the ring of symmetric tensors of $\otimes^n_A A[X]$ to A, which has the following property. For every $f(X)$ in $A[X]$ the residue class of $f(X)$ modulo the ideal $(F(X))$ gives by multiplication an A-linear endomorphism $\mu_F(f)$ on the free A-module $A[X]/(F(X))$. In Theorem (2.4) it is shown that $u_F(f(X) \otimes \cdots \otimes f(X)) = \det(\mu_F(f))$.

Theorem (2.4) is the technical heart of the present paper and gives a nice relation between the symmetric operators with the coefficients of the characteristic polynomial of $\mu_F(f)$. A relation of the homomorphism u_F to resultants is touched upon in Section (3).

By studying the properties of such homomorphisms u_F we classify in Section (4), the set of ideals $I \subseteq A[X]_U$ such that the residue ring $A[X]_U/I$ is free as an A-module.

We denote with $\otimes^{(n)}_A A[X]$ the ring of invariant tensors of $\otimes^n_A A[X]$ under the standard action of the symmetric group in n-letters. In Section (5) we show that the functor parameterizing ideals of $A[X]_U$ such that the residue rings are free of rank n as A-modules is represented by a fraction ring of $\otimes^{(n)}_A A[X]$. A fraction ring which we show in Section (6) is isomorphic to the ring of symmetric tensors $\otimes^{(n)}_A A[X]_U$.

Then finally we summarize the accumulated results in the main Theorem which states that the n-fold symmetric product $\mathrm{Symm}^n_A(A[X]_U)$ represents the functor $\mathcal{H}ilb^n_{A[X]_U/A}$, and where the universal family is given as $\mathrm{Symm}^{n-1}_A(A[X]_U) \times_A \mathrm{Spec}(A[X]_U)$.

As an application of our result we have that the Hilbert scheme of n-points on the affine line $\mathbf{A}^1_A$ over $\mathrm{Spec}\, A$ is represented by the affine n-space $\mathbf{A}^n_A$. Here A is any commutative unitary ring. Furthermore we have that $\mathcal{H}ilb^1_{A[X]_U/A}$, the Hilbert functor parameterizing 1 point on $\mathrm{Spec}(A[X]_U)$ is represented by the scheme $\mathrm{Spec}(A[X]_U)$, [10]. In the case when the base ring $A = k$ is a field, we have that the ring of symmetric tensors of $\otimes^n_k k[X]_U$ parameterizes the closed subschemes of the line $\mathrm{Spec}(k[X])$ having length n and support in the subset of the line corresponding to the prime ideals $\{P \subset k[X] \mid P \cap U = \emptyset\}$. In particular we recover the situation considered in [13] where $k[X]_U = k[X]_{(X)}$ is the local ring of the maximal ideal $(X) \subseteq k[X]$. A more precise discussion of some applications of Theorem (8.2) is given in the end of Section (8).

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§2. - Symmetric operators on the polynomial ring.

Given a monic polynomial $F(X)$ in $A[X]$ which has positive degree n. Then the residue ring $A[X]/(F(X))$ is a free A-module of rank n. Any element $f(X)$ in $A[X]$ gives by multiplication by the residue class of $f(X)$ modulo the ideal $(F(X))$, an A-linear endomorphism $\mu_F(f)$ on $A[X]/(F(X))$. The goal of Section (2) is the

construction of an A-algebra homomorphism $u_F : \otimes_A^{(n)} A[X] \to A$, where $\otimes_A^{(n)} A[X]$ is the ring of symmetric functions, with the property that for any $f(X)$ in $A[X]$ we have that

$$u_F(f(X) \otimes \cdots \otimes f(X)) = \det(\mu_F(f)).$$

2.1. The symmetric operators. We recall the symmetric operators which were introduced in [13]. Let $A[X]$ denote the ring of polynomials in the variable X over a commutative ring A. We write $\otimes_A^n A[X]$ for the tensor algebra $A[X] \otimes_A \cdots \otimes_A A[X]$ (n-copies of $A[X]$). To every element $f(X)$ in $A[X]$ we let $f(X_i) = 1 \otimes \cdots \otimes 1 \otimes f(X) \otimes 1 \otimes \cdots \otimes 1$ in $\otimes_A^n A[X]$, where the $f(X)$ occurs at the i'th place. We identify $X = 1 \otimes \ldots 1 \otimes X$ in $\otimes_A^{n+1} A[X]$, and we consider $\otimes_A^{n+1} A[X]$ as the polynomial ring in the variable X over $\otimes_A^n A[X]$.

To each element $f(X)$ in $A[X]$ and for every positive integer n, we define the symmetric tensors $s_{1,n}(f(X)), \ldots, s_{n,n}(f(X))$ by the following identity in $\otimes_A^{n+1} A[X]$.

$$\begin{aligned} \Delta_{n,f(X)}(X) &= \prod_{i=1}^{n} (X - f(X_i)) \\ &= X^n - s_{1,n}(f(X)) X^{n-1} + \cdots + (-1)^n s_{n,n}(f(X)). \end{aligned} \tag{2.1.1}$$

We have that $s_{1,n}(X), \ldots, s_{n,n}(X)$ are the elementary symmetric functions in $X_1, \ldots, X_n$. We denote the ring of symmetric tensors of $\otimes_A^n A[X]$, which is the polynomial ring in the variables $s_{1,n}(X), \ldots, s_{n,n}(X)$ over A, by $\otimes_A^{(n)} A[X]$.

The element $\Delta_{n,f(X)}(X)$ is a polynomial in the variable X, having coefficients in the ring of symmetric functions $\otimes_A^{(n)} A[X]$. Since $\Delta_{n,f(X)}(X)$ is a monic polynomial of degree n, we have that the residue ring

$$V_{n,f(X)} = \big(\otimes_A^{(n)} A[X] \otimes_A A[X] \big) / (\Delta_{n,f(X)}(X)) \tag{2.1.2}$$

is a free $\otimes_A^{(n)} A[X]$-module of rank n.

2.2. The homomorphism u_F. Let $F(X) = X^n - u_1 X^{n-1} + \cdots + (-1)^n u_n$ be a monic polynomial in the polynomial ring $A[X]$. There is a unique A-algebra homomorphism

$$u_F : \otimes_A^{(n)} A[X] \to A$$

determined by $u_F(s_{i,n}(X)) = u_i$. The homomorphism u_F gives $A[X]$ an $\otimes_A^{(n)} A[X]$-module structure and we have a natural identification

$$V_{n,X} \otimes_{\otimes_A^{(n)} A[X]} A \cong A[X]/(F(X)).$$

Theorem 2.3. *Let M be a quadratic matrix, having coefficients in a commutative ring A. Assume that the characteristic polynomial $P_M(X) = \prod_{i=1}^n (X - a_i)$ of M splits into linear factors over A. Then we have for any polynomial $f(X)$ in $A[X]$ that the matrix $f(M)$ has characteristic polynomial $P_{f(M)}(X) = \prod_{i=1}^n (X - f(a_i))$.*

Proof. The result is well known when A is a field. A proof of the Spectral Theorem over general commutative rings is found in [14].

Theorem 2.4. *Let $F(X)$ be a non-constant, monic polynomial in $A[X]$. Denote with n the degree of $F(X)$. For any element $f(X)$ in $A[X]$ we let $\mu_F(f)$ be the A-linear endomorphism on $A[X]/(F(X))$ given as multiplication by the residue class of $f(X)$ modulo the ideal $(F(X))$. We have that the characteristic polynomial of $\mu_F(f)$ is*

$$X^n - u_F(s_{1,n}(f(X)))X^{n-1} + \cdots + (-1)^n u_F(s_{n,n}(f(X))).$$

In particular we have that $u_F(f(X) \otimes \cdots \otimes f(X))$ is the determinant of $\mu_F(f)$.

Proof. For any $f(X)$ in $A[X]$ we let $\mu(f)$ be the $\otimes_A^{(n)} A[X]$-linear endomorphism on $V_{n,X} = \otimes_A^{(n)} A[X] \otimes_A A[X]/(\Delta_{n,X}(X))$ given as multiplication by the residue class of $f(X)$ in $V_{n,X}$. Let $u_F : \otimes_A^{(n)} A[X] \to A$ be the A-algebra homomorphism determined by $F(X)$ in $A[X]$. We have that the induced A-linear endomorphism $\mu(f) \otimes \mathrm{id}$ on $V_{n,X} \otimes_{\otimes_A^{(n)} A[X]} A \cong A[X]/(F(X))$ is $\mu_F(f)$. Hence to prove our Theorem it suffices to show that the endomorphism $\mu(f)$ has characteristic polynomial

$$X^n - s_{1,n}(f(X))X^{n-1} + \cdots + (-1)^n s_{n,n}(f(X)). \tag{2.4.1}$$

To show that $\mu(f)$ has characteristic polynomial (2.4.1) I claim that it is sufficient to show that $\mu(X)$ has characteristic polynomial $\Delta_{n,X}(X)$. Indeed, we have that $\otimes_A^{(n)} A[X]$ is a subring of $\otimes_A^n A[X]$. Hence by ring extension, we consider $\mu(f)$ as an $\otimes_A^n A[X]$-linear endomorphism on

$$V_{n,X} \otimes_{\otimes_A^{(n)} A[X]} \otimes_A^n A[X] \cong \otimes_A^{n+1} A[X]/(\Delta_{n,X}(X)). \tag{2.4.2}$$

We have that that $\Delta_{n,X}(X) = \prod_{i=1}^n (X - X_i)$ splits into linear factors over $\otimes_A^n A[X]$. Thus if $\mu(X)$ has characteristic polynomial $\Delta_{n,X}(X)$ then it follows from Theorem (2.3) that $\mu(f)$ has characteristic polynomial $\Delta_{n,f(X)}(X) = \prod_{i=1}^n (X - f(X_i))$. We have that $\Delta_{n,f(X)}(X)$ written out in terms of symmetric functions is (2.4.1). Thus what remains is to show that the endomorphism $\mu(X)$ on $\otimes_A^{n+1} A[X]/(\Delta_{n,X}(X)$ has characteristic polynomial $\Delta_{n,X}(X)$.

Let x^i be the residue class of X^i modulo the ideal $(\Delta_{n,X}(X))$ in $\otimes_A^{n+1} A[X]$. We have that $1, x, \dots, x^{n-1}$ form a $\otimes_A^n A[X]$ basis for (2.4.2). The matrix M representing the endomorphism $\mu(X)$, with respect to the given basis is easy to describe and is called the *companion matrix* of $\Delta_{n,X}(X)$. In general, if $F(X) = X^n - u_1 X^{n-1} - \cdots - u_n$ is a monic polynomial, then the companion matrix of $F(X)$ is the matrix

$$M_F = \begin{pmatrix} 0 & 0 & \dots & 0 & u_n \\ 1 & 0 & & \vdots & u_{n-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 1 & u_1 \end{pmatrix}.$$

Note that the matrix obtained by deleting first row and first column of M_F, is the companion matrix of $G(X) = X^{n-1} - u_1 X^{n-2} - \cdots - u_{n-1}$. It follows readily by induction on the size n of M_F, that the determinant $\det(XI - M_F) = F(X)$. Thus we get that the matrix M representing the endomorphism $\mu(X)$ with respect to the basis $1, x, \dots, x^{n-1}$ has characteristic polynomial $\Delta_{n,X}(X)$. We have proven the Theorem.

§3. - The Norm function and resultants.

We will use the notation from the preceding sections. Let $\varphi : A \to K$ be an ring homomorphism. For any $f(X) = a_m X^m + \cdots + a_m$ in $A[X]$, we write $f^\varphi(X) = \varphi(a_m)X^m + \cdots + \varphi(a_m)$ in $K \otimes_A A[X] = K[X]$. Thus $f^\varphi(X)$ is the reduction of $f(X)$ modulo the kernel of φ.

3.1. Definition. Given a monic polynomial $F(X)$ in $A[X]$ which has positive degree n. We define the *Norm function* $N_F : A[X] \to A$ with respect to $F(X)$, by sending $f(X)$ in $A[X]$ to

$$N_F(f(X)) := u_F(f(X) \otimes \cdots \otimes f(X)), \tag{3.1.1}$$

where $u_F : \otimes_A^{(n)} A[X] \to A$ is the A-algebra homomorphism (2.2) determined by $F(X)$. Let $\mu_F(f)$ be the A-linear endomorphism on $A[X]/(F(X))$ given as multiplication by the residue class of $f(X)$ modulo the ideal $(F(X))$. By Theorem (2.4) we have that $N_F(f)$ is the determinant of the endomorphism $\mu_F(f)$. We say that $N_F(f(X))$ is the *norm* of $f(X)$ with respect to $F(X)$.

Let $\varphi : A \to K$ be an A-algebra homomorphism. If $F(X)$ is a monic polynomial of degree n in $A[X]$, then $F^\varphi(X)$ is a monic polynomial of degree n in $K[X]$. Therefore $F^\varphi(X)$ determines a K-algebra homomorphism $u_{F^\varphi} : \otimes_K^{(n)} K[X] \to K$. We have the following relation between the norm function with respect to $F(X)$ and the norm function with respect to $F^\varphi(X)$.

Lemma 3.2. *Let $\varphi : A \to K$ be an A-algebra homomorphism. Let $F(X)$ in $A[X]$ be a monic polynomial of positive degree n. Then for all $f(X)$ in $A[X]$ we have for each $i = 1, \dots, n$ that $\varphi \circ u_F(s_{i,n}(f(X))) = u_{F^\varphi}(s_{i,n}(f^\varphi(X)))$. In particular we have that $\varphi(N_F(f(X))) = N_{F^\varphi}(f^\varphi(X))$.*

Proof. The A-algebra homomorphism $\varphi : A \to K$ induces by base change a homomorphism $\hat{\varphi} : \otimes_A^{(n)} A[X] \to \otimes_K^{(n)} K[X]$, which maps $s_{i,n}(f(X))$ to $s_{i,n}(f^\varphi(X))$ for each $i = 1, \dots, n$ and for each $f(X)$ in $A[X]$. Furthermore it is clear that $\varphi \circ u_F = u_{F^\varphi} \circ \hat{\varphi}$. where u_{F^φ} is the K-algebra homomorphism $u_{F^\varphi} : \otimes_K^{(n)} K[X] \to K$ determined by the monic polynomial $F^\varphi(X) \in K[X]$. We have proven the Lemma.

Proposition 3.3. *Let $P(X)$ and $Q(X)$ be two monic polynomials in $A[X]$. Let p be the degree of $P(X)$ and let q be the degree of $Q(X)$. Assume that both p and q are positive. Then we have that $N_P(Q(X)) = (-1)^{pq} N_Q(P(X))$.*

Proof. Let $X_i = 1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \cdots \otimes 1$ in the ring $\otimes_A^{p+q} A[X]$, where the X occurs at the i'th place. We consider the following product

$$\operatorname{res}(p, q) = \prod_{i=1}^{p} \prod_{j=1}^{q} (X_i - X_{p+j}). \tag{3.3.1}$$

It follows from (3.3.1) that the product $\operatorname{res}(p, q)$ is symmetric in $X_1, \dots, X_p$ and symmetric in $X_{p+1}, \dots, X_{p+q}$. Thus $\operatorname{res}(p, q)$ is an element of $\big(\otimes_A^{(p)} A[X]\big) \otimes_A \big(\otimes_A^{(q)} A[X]\big)$. The monic polynomial $P(X)$ in $A[X]$ determines an A-algebra homomorphism (2.2) $u_P : \otimes_A^{(p)} A[X] \to A$. We let $\hat{u}_P : \big(\otimes_A^{(p)} A[X]\big) \otimes_A \big(\otimes_A^{(q)} A[X]\big) \to$ $\otimes_A^{(q)} A[X]$ be the induced map. Similarily we let $\hat{u}_Q : \big(\otimes_A^{(p)} A[X]\big) \otimes_A \big(\otimes_A^{(q)} A[X]\big) \to$

$\otimes_A^{(p)} A[X]$ be the map induced by $u_Q : \otimes_A^{(q)} A[X] \to A$. Clearly we have that $u_P \circ \hat{u}_Q = u_Q \circ \hat{u}_P$.

For fixed i we have that

$$\prod_{j=1}^{q}(X_i - X_{p+j}) = X_i^q - s_{1,q}(X)X_i^{q-1} + \cdots + (-1)^q s_{q,q}(X). \tag{3.3.2}$$

It follows from (3.3.2) and the definition of the homomorphism u_Q that $\hat{u}_Q$ maps $\mathrm{res}(p,q)$ to $\prod_{i=1}^{p}(Q(X_i)) = Q(X)\otimes\cdots\otimes Q(X)$ in $\otimes_A^{(p)} A[X]$. Similarly we get that $\mathrm{res}(q,p) = (-1)^{pq}\,\mathrm{res}(p,q)$ is mapped to $\prod_{j=1}^{q} P(X_j)$ in $\otimes_A^{(q)} A[X]$ by $\hat{u}_P$. Thus we have that

$$\begin{aligned} N_P(Q(X)) &= u_P(Q(X)\otimes\cdots\otimes Q(X)) = u_P(\hat{u}_Q(\mathrm{res}(p,q)) \\ &= u_Q(\hat{u}_P(\mathrm{res}(p,q))) = u_Q(\hat{u}_P((-1)^{pq}\,\mathrm{res}(q,p))) \\ &= (-1)^{pq}u_Q(P(X)\otimes\cdots\otimes P(X)) = (-1)^{pq}N_Q(P(X)), \end{aligned}$$

proving our claim.

Remark. When A is a field we have that Proposition (3.3) is a well-known formula for resultants.

Remark. The trick which we use in the proof of the Proposition is to consider the product $\mathrm{res}(p,q)$ (3.3.1), which we found in [11] (Chapter IV, §8, the proof of Proposition (8.3), pp. 202-203).

§4. - Residues of fraction rings.

In this section we investigate and describe ideals I in fraction rings $A[X]_U$, such that the residue ring $A[X]_U/I$ is a free A-module of rank n.

The key point is Theorem (4.2) below which generalizes the Main theorem of [13] (Theorem (2.3), Assertions (4), (5) and (2)). In [13] A was an algebra defined over some field k and the multiplicatively closed set $U \subseteq A[X]$ was the set of $f(X)$ in $k[X]$ such that $f(0) \neq 0$. Having established Theorem (4.2) the other results of Section (4) follows, *mutatis mutandis*, from [13]. We have included proofs in order to make the paper self contained.

4.1. Notation. We fix a ring A and a multiplicatively closed subset $U \subseteq A[X]$ of the polynomial ring $A[X]$. We write the fraction ring of $A[X]$ with respect to U as $A[X]_U$.

Let $\varphi : A \to K$ be an A-algebra homomorphism. We denote by $U^\varphi \subseteq K[X]$ the image of $U \subseteq A[X]$ under the induced map $A[X] \to K[X]$. If $f(X)$ is an element in $A[X]$ we denote with $f^\varphi(X)$ the polynomial in $K[X]$ obtained by applying φ to the coefficients of $f(X)$. We have that $U^\varphi \subseteq K[X]$ is the set of polynomials $f^\varphi(X)$ where $f(X)$ is in $U \subseteq A[X]$

Theorem 4.2. *Given a monic polynomial $F(X)$ in $A[X]$ which has positive degree n. Let $U \subseteq A[X]$ be a multiplicatively closed subset. The following three assertions are equivalent.*

(1) *The canonical map $A[X]/(F(X)) \to A[X]_U/(F(X))$ is an isomorphism.*
(2) *The residue classes of $1, X, \ldots, X^{n-1}$ modulo $F(X)A[X]_U$ form a basis for the A-module $A[X]_U/(F(X))$.*
(3) *The norm $N_F(f(X))$ with respect to $F(X)$, is a unit in A for all $f(X)$ in $U \subseteq A[X]$.*

Proof. It is clear that the two first assertions are equivalent. We will show that Assertion (1) is equivalent with Assertion (3). The fraction map $A[X]/(F(X)) \to A[X]_U/(F(X))$ is an isomorphism if and only if the class of $f(X)$ in $A[X]/(F(X))$ is invertible for all $f(X)$ in the multiplicatively closed set $U \subseteq A[X]$. The residue class of $f(X)$ modulo $(F(X)) \subseteq A[X]$ is invertible if and only if the endomorphism $\mu_F(f)$ on $A[X]/(F(X))$ given as multiplication by the residue class of $f(X)$ is invertible. By Theorem (2.4) we have that the determinant of the endomorphism $\mu_F(f)$ is the norm $N_F(f(X))$, and our claim follows.

Corollary 4.3. *Let A be a local ring. Denote the residue field of A as K. Assume that the K-vector space $A[X]_U/(F(X)) \otimes_A K$ has a basis given by the residue classes of $1, X, \dots, X^{n-1}$. Then we have that the A-module $A[X]_U/(F(X))$ has a basis given by the residue classes of $1, X, \dots, X^{n-1}$. In particular we have that $A[X]_U/(F(X))$ is a finitely generated A-module.*

Proof. Let $\varphi : A \to K$ be the residue class map. We have a canonical isomorphism

$$A[X]_U/(F(X)) \otimes_A K \cong K[X]_{U^\varphi}/(F^\varphi(X)). \tag{4.3.1}$$

By assumption we have that the residue classes of $1, X, \dots, X^{n-1}$ is a K-basis for (4.3.1). It then follows from Assertion (3) of Theorem (4.2) that the norm $N_{F^\varphi}(f^\varphi(X))$ with respect to $F^\varphi(X)$, is a unit in K for all $f^\varphi(X)$ in $U^\varphi \subseteq K[X]$. Furthermore we have by Lemma (3.2) that $N_{F^\varphi}(f^\varphi(X)) = \varphi(N_F(f(X)))$. Hence $N_F(f(X))$ is a unit in A for all $f(X)$ in $U \subseteq A[X]$. Now the claim follows by Assertion (3) of Theorem (4.2).

Remark. It is not true that A-modules $A[X]_U/(F(X))$, with monic polynomials $F(X) \in A[X]$, in general are finitely generated. For instance, let $A = k[T]$ be the polynomial ring in the variable T over a field k. Let $U \subseteq A[X]$ be the set of non-zero elements of A, thus U is the set of non-zero polynomials $f(T)$ in $k[T]$. Let $F(X) = X - T$, which is a monic polynomial in $A[X]$ of degree 1. We have that $A[X]/(F(X)) = A$, from which it follows that $A[X]_U/(F(X)) = k(T)$, the function field of $A = k[T]$, clearly not finitely generated over $A = k[T]$.

Lemma 4.4. *Let $A = K$ be a field, and let $U \subseteq K[X]$ be a multiplicatively closed subset. Let $F(X)$ be a polynomial in $K[X]$. Then we have that the K-vector space $K[X]_U/(F(X))$ is generated by the residue classes of $1, X, \dots, X^{n-1}$ modulo $F(X)K[X]_U$, and where n is the degree of $F(X)$.*

Proof. Denote by x^i the residue class of X^i modulo the ideal $F(X)K[X]_U$. Taking fraction commutes with taking quotients. Hence given $f(X)$ in $K[X]$ it is clear that the class of $f(X)$ in $K[X]_U/(F(X))$ is in the span of $1, x, \dots, x^{n-1}$. We must show that given $f(X)$ in $U \subseteq K[X]$ then $f(x)^{-1}$, the residue class of $f(X)^{-1}$ modulo $F(X)K[X]_U$, can be written as a K-linear combination of $1, x, \dots, x^{n-1}$.

Let $f(X)$ be an element of $U \subseteq K[X]$, which we may assume to be irreducible. Indeed, if $f(X)$ is invertible in $K[X]_U$, then it follows that the irreducible factors of $f(X)$ are invertible in $K[X]_U$. Hence it suffices to show that the irreducible factors of $f(X)$ are in the K-linear span of $1, x, \dots, x^{n-1}$. Consequently we assume that the ideal $(f(X))$ in $K[X]$ is maximal.

If $F(X)$ is in the ideal $(f(X))$ then $F(X) = f(X)G(X)$, where $\deg(G(X)) < \deg(F(X))$. We have that $G(X)$ and $F(X)$ generate the same ideal in $K[X]_U$. Thus by replacing $F(X)$ by $G(X)$, we may assume that $F(X)$ is not in the ideal $(f(X))$.

That is the ideals $(F(X))$ and $(f(X))$ are coprime. Hence there exist polynomials $h(X)$ and $H(X)$ such that $f(X)h(X)+F(X)H(X)=1$ in $K[X]$. It then follows that

$$h(X)+f(X)^{-1}F(X)H(X)=f(X)^{-1},$$

in $K[X]_U$. From which we get that $h(x)=f(x)^{-1}$ in $K[X]_U/(F(X))$. The element $h(X)$ is in $K[X]$, hence its residue class $f(x)^{-1}$ in $K[X]_U/(F(X))$, is in the span of $1,x,\dots,x^{n-1}$. We have proven our claim.

Theorem 4.5. *Given an ideal $I\subseteq A[X]_U$ such that the residue ring $A[X]_U/I$ is a free A-module of rank n. Then there exists a unique monic polynomial $F(X)$ in $A[X]$ of degree n, whose image in $A[X]_U$ generates the ideal I. In particular we have that the canonical map $A[X]/(F(X))\to A[X]_U/I$ is an isomorphism*

Proof. Let $V=\oplus_{i=1}^n A$. By assumption we have that $V=A[X]_U/I$ as A-modules. The A-module V is also an $A[X]$-module by the natural map $A[X]\to A[X]_U/I$. Thus the variable X is mapped to an A-linear endomorphism θ on V. Let $F(X)=F_\theta(X)$ be the characteristic polynomial of the endomorphism θ.

By the Cayley-Hamilton Theorem (see e.g [11], XIV §3, Theorem (3.1), p. 561), any endomorphism satisfies its characteristic polynomial. Hence we have that $F(X)$ is in the kernel of the natural map $A[X]\to A[X]_U/I$. We have thus shown that

$$A[X]_U/(F(X))\to A[X]_U/I \tag{4.5.1}$$

is surjective. We claim that the map (4.5.1) is an isomorphism of A-modules.

To show that (4.5.1) is an isomorphism of A-modules it is sufficient to show (4.5.1) when A is a local ring. Let A be a local ring. Let $\varphi : A\to K$ be the residue class map. Since the map in (4.5.1) is surjective it follows that the K-vector space $A[X]_U/(F(X))\otimes_A K$ is of dimension greater or equal to n, the rank of $A[X]_U/I$. We have that $A[X]_U/(F(X))\otimes_A K$ is isomorphic to $K[X]_{U^\varphi}/(F^\varphi(X))$. By Lemma (4.4) we have that the classes of $1,X,\dots,X^{n-1}$ generate the K-vector space $A[X]_U/(F(X))\otimes_A K$. Since the dimension of $A[X]_U/(F(X))\otimes_A K$ is at least n it follows that the dimension equals n. It then follows from Corollary (4.3) that $A[X]_U/(F(X))$ is a free A-module where the classes of $1,X,\dots,X^{n-1}$ form a basis. Thus (4.5.1) is an isomorphism.

We have that the classes of $1,X,\dots,X^{n-1}$ form a basis for the A-module $A[X]_U/(F(X))$. It follows from Theorem (4.2) that $A[X]/(F(X))$ is canonically isomorphic to $A[X]_U/(F(X))$. We have proven the Theorem.

§5. - Parameterizing ideals of fraction rings.

In Section (5) we will extract the results from the preceding section and show that there exist a ring parameterizing the set of ideals in $A[X]_U$ such that the residue ring is a free, rank n module over A.

5.1. The functor $\mathcal{F}_U^n$. We fix a multiplicatively closed subset $U\subseteq A[X]$. For any A-algebra K we let $\mathcal{F}_U^n(K)$ denote the set of residue rings $K\otimes_A A[X]_U/I$, which are free and of rank n as K-modules. The map sending an A-algebra K to the set $\mathcal{F}_U^n(K)$ becomes in a natural way a covariant functor from the category of A-algebras to sets.

The A-valued points of $\mathcal{F}_U^n$, that is the set $\mathcal{F}_U^n(A)$, correspond to residues of $A[X]_U$ which are free and of rank n as A-modules. We call $\mathcal{F}_U^n$ the functor parameterizing the free, rank n residues of $A[X]_U$. The goal of Section (5) is to show that the functor $\mathcal{F}_U^n$ is representable.

5.2. The construction of the universal objects. To a multiplicatively closed subset $U \subseteq A[X]$ and a positive integer n, we define the multiplicatively closed subset $U(n) \subseteq \otimes_A^{(n)} A[X]$ as

$$U(n) = \{f(X) \otimes \cdots \otimes f(X) \mid f(X) \in U \subseteq A[X]\}. \tag{5.2.1}$$

Let $\xi_U : \otimes_A^{(n)} A[X] \to (\otimes_A^{(n)} A[X])_{U(n)}$ denote the fraction map. Recall (2.3.1) that we have defined $\Delta_{n,X}(X) = X^n - s_{1,n}(X)X^{n-1} + \cdots + (-1)^n s_{n,n}(X)$ in the polynomial ring in the variable X over the ring of symmetric functions $\otimes_A^{(n)} A[X]$. We denote by $\Delta_{n,X}^{\xi}(X)$ the element in $(\otimes_A^{(n)} A[X])_{U(n)} \otimes_A A[X]$ obtained by applying ξ_U to the coefficients of $\Delta_{n,X}(X)$. Furthermore we define

$$\begin{aligned} V_{n,X}^U &= V_{n,X} \otimes_{\otimes_A^{(n)} A[X]} (\otimes_A^{(n)} A[X])_{U(n)} \\ &= (\otimes_A^{(n)} A[X])_{U(n)} \otimes_A A[X]/(\Delta_{n,X}^{\xi}(X)). \end{aligned} \tag{5.2.2}$$

We have that $V_{n,X}^U$ is a free $(\otimes_A^{(n)} A[X])_{U(n)}$-module of rank n. Note that $V_{n,X}^U$ is not an $(\otimes_A^{(n)} A[X])_{U(n)}$-valued point of the functor $\mathcal{F}_U^n$ since $V_{n,X}^U$ is not a residue of $(\otimes_A^{(n)} A[X])_{U(n)} \otimes_A A[X]_U$, but only a residue of $(\otimes_A^{(n)} A[X])_{U(n)} \otimes_A A[X]$. We need to consider what happens with $V_{n,X}^U$ when localized, as an $A[X]$-algebra, in the multiplicatively closed set $U \subseteq A[X]$.

Lemma 5.3. *Let $U \subseteq A[X]$ be a multiplicatively closed subset. We have that the fraction map of $A[X]$-modules*

$$V_{n,X}^U \to \big(\otimes_A^{(n)} A[X]\big)_{U(n)} \otimes_A A[X]_U/(\Delta_{n,X}^{\xi_U}(X)),$$

obtained by localization with respect to $U \subseteq A[X]$, is an isomorphism.

Proof. Let $f(X)$ be an element of $A[X]$. Let $\mu(f)$ be the $\otimes_A^{(n)} A[X]$-linear endomorphism on $V_{n,X}$, given as multiplication by the class of $f(X)$ in $V_{n,X}$. It follows by Theorem (2.4) with $u_F = \mathrm{id}$ the identity morphism, that the determinant of $\mu(f)$ is $f(X) \otimes \cdots \otimes f(X)$. We then have that the determinant of the induced endomorphism $\mu(f) \otimes \mathrm{id}$ on $V_{n,X}^U$ is $\xi_U(f(X) \otimes \cdots \otimes f(X))$. Thus for all $f(X)$ in $U \subseteq A[X]$, we have that the determinant of $\mu(f) \otimes \mathrm{id}$ is a unit in $(\otimes_A^{(n)} A[X])_{U(n)}$. That is, the class of $f(X)$ in $V_{n,X}^U$ is invertible, for all $f(X)$ in $U \subseteq A[X]$. It follows that the fraction map of the Lemma is an isomorphism.

Theorem 5.4. *Let $U \subseteq A[X]$ be a multiplicatively closed subset. For every positive integer n we have that the functor $\mathcal{F}_U^n$ parameterizing free, rank n residues of $A[X]_U$, is represented by the A-algebra $\big(\otimes_A^{(n)} A[X]\big)_{U(n)}$. The universal element is given by the residue ring $V_{n,X}^U$.*

Proof. We have that $V_{n,X}^U$ is a free $F = (\otimes_A^{(n)} A[X])_{U(n)}$-module of rank n. It follows by Lemma (5.3) that $V_{n,X}^U$ is an element of the set $\mathcal{F}_U^n(F)$. Consequently we have a natural transformation $\Phi : \mathrm{Hom}_{A\text{-alg}}(F, -) \to \mathcal{F}_U^n$, which, for any A-algebra K, sends an A-algebra homomorphism $F \to K$ to the free K-module $V_{n,X}^U \otimes_F K$. We need to show that Φ is an isomorphism.

Let $\varphi : A \to K$ be an A-algebra homomorphism. Let $I \subseteq K \otimes_A A[X]_U$ be an ideal such such that the residue ring $K \otimes_A A[X]_U/I$ is a free K-module of rank n. We have that $K \otimes_A A[X]_U \cong K[X]_{U^\varphi}$, where $U^\varphi \subseteq K[X]$ is the image of $U \subseteq A[X]$ by the homomorphism $A[X] \to K[X]$. Hence it follows from Theorem (4.5) that there exist a unique monic polynomial $F(X)$ in $K[X]$ of degree n, such that the image of $F(X)$ in the fraction ring $K[X]_{U^\varphi}$ generates I. Moreover, we have that the canonical fraction map

$$K[X]/(F(X)) \to K[X]_{U^\varphi}/(F(X)) = K \otimes_A A[X]_U/I \tag{5.4.1}$$

is an isomorphism. Let $F(X) = X^n - u_1 X^{n-1} + \ldots (-1)^n u^n$ in $K[X]$.

We get by base change an A-algebra homomorphism $\hat{\varphi} : \otimes_A^{(n)} A[X] \to \otimes_K^{(n)} K[X]$, where $\varphi : A \to K$ is the structure map. The coefficients of $F(X)$ determine a K-algebra homomorphism $u_F : \otimes_K^{(n)} K[X] \to K$ by $u_F(s_{i,n}(X)) = u_i$ for each $i = 1, \ldots, n$. It is clear that the composite morphism $\hat{\varphi} \circ u_F$ is an A-algebra homomorphism such that

$$V_{n,X} \otimes_{\otimes_A^{(n)} A[X]} K = K[X]/(F(X)).$$

We need to show that the composite map $\hat{\varphi} \circ u_F$ factors through the fraction ring F of $\otimes_A^{(n)} A[X]$. Since the fraction map of (5.4.1) is an isomorphism it follows by Assertion (3) of Theorem (4.3), that $N_F(f^\varphi(X))$ is a unit in K for all $f^\varphi(X) \in U^\varphi \subseteq K[X]$. It follows that the symmetric functions $f(X) \otimes \cdots \otimes f(X)$ in $\otimes_A^{(n)} A[X]$ for all $f(X) \in U \subseteq A[X]$, are mapped to units in K by the homomorphism $\hat{\varphi} \circ u_F$. By the universal property of the fraction ring F, the A-algebra homomorphism $\hat{\varphi} \circ u_F$ factors through F.

Thus we have shown that for any A-algebra K, and to any element V in $\mathcal{F}_U^n(K)$ there exist an A-algebra homomorphism $F \to K$ such that $V = V_{n,X}^U \otimes_F K$.

To complete the proof of the Theorem we need to show that two different elements in $\mathrm{Hom}_{A\text{-alg}}(F, K)$ correspond to two different elements in $\mathcal{F}_U^n(K)$. Let $\rho : F \to K$ be an A-algebra homomorphism. Since F is a fraction ring of $\otimes_A^{(n)} A[X]$, it follows that ρ is determined by its action on the elementary symmetric functions $s_{1,n}(X), \ldots, s_{n,n}(X)$. Let $G(X) = X^n - v_1 X^{n-1} + \cdots + (-1)^n v_n$, where $v_i = \rho(s_{i,n}(X))$, for each $i = 1, \ldots, n$. We have that that $V_{n,X}^U \otimes_F K$ is isomorphic to $K[X]_{U^\varphi}/(G(X))$, which by Theorem (4.5) is isomorphic to $K[X]/(G(X))$. It is clear that two polynomials $F(X)$ and $G(X)$ in $K[X]$ of degree n and with leading coefficient 1, generate the same ideal if and only if $F(X) = G(X)$. Hence it follows that two different A-algebra homomorphisms $F \to K$ give two different ideals in $K \otimes_A A[X]_U$. We have proven the Theorem.

§**6. - Symmetric products of fraction rings.**

In the next two sections we study more in detail the ring $(\otimes_A^{(n)} A[X])_{U(n)}$ and the ring $V_{n,X}^U$ which together form the universal pair of Theorem (5.4).

In Section (6) we will show that the fraction ring $(\otimes_A^{(n)} A[X])_{U(n)}$ is canonically isomorphic to the ring symmetric tensors of $\otimes_A^n A[X]_U$. In Section (7) we show that $V_{n,X}^U$ is isomorphic to $(\otimes_A^{(n)} A[X])$ ... $\otimes$ ... $A[X]$

Proposition 6.1. *Let K and L be two A-algebras. Let $U \subseteq K$ and $V \subseteq L$ be multiplicatively closed subsets. Then we have for any K-module M and any L-module N a canonical isomorphism of A-modules $M_U \otimes_A N_V \cong (M \otimes_A N)_{U \cdot V}$, where $U \cdot V \subseteq K \otimes_A L$ is the multiplicatively closed subset given as*

$$U \cdot V = \{f \otimes g \mid f \in U \subseteq K, g \in V \subseteq L\}.$$

In particular we have that $K_U \otimes_A L_V \cong (K \otimes_A L)_{U \cdot V}$.

Proof. We have by definition that $M_U = M \otimes_K K_U$. Hence our claim follows if we show that $K_U \otimes_A L_V \cong (K \otimes_A L)_{U \cdot V}$.

We have a natural, well defined map $K \otimes_A L \to K_U \otimes_A L_V$, sending $\sum_{i=1}^m (f_i \otimes g_i)$ to $\sum_{i=1}^m ((f_i, 1) \otimes (g_i, 1))$. The map $K \otimes_A L \to K_U \otimes_A L_V$ maps an element $(u \otimes v)$ in $U \cdot V$ to a unit in $K_U \otimes_A L_V$. Hence by the universal property of fraction rings we get a map

$$(K \otimes_A L)_{U \cdot V} \to K_U \otimes_A L_V. \tag{6.1.1}$$

The composite map $K \to K \otimes_A L \to (K \otimes_A L)_{U \cdot V}$ sends any element u in $U \subseteq K$ to the unit $u \otimes 1$ in $(K \otimes_A L)_{U \cdot V}$. Hence we get a natural A-algebra homomorphism $K_U \to (K \otimes_A L)_{U \cdot V}$. Similarily we get a map $L_V \to (K \otimes_A L)_{U \cdot V}$. It follows from the universal property of the tensor product that we have get a map $K_U \otimes_A L_V \to (K \otimes_A L)_{U \cdot V}$, easily seen to be the inverse of (6.1.1).

Lemma 6.2. *Let U be a multiplicatively closed subset of a ring A, containing the identity element. Assume that U is a subset of a multiplicatively closed set V, such that for any element v in V there exist an element w in V such that vw is in U. Then we have that A_U is canonical isomorphic to A_V.*

Proof. We have by assumption that $U \subseteq V$. Hence there is a canonical map $A_U \to A_V$, which is easily seen to be both injective and surjective.

6.3. An action of the symmetric group. Let K be an A-algebra. The symmetric group of n-letters $\mathfrak{S}_n$ acts on the tensor algebra $\otimes_A^n K$ in a natural way. If $\sum_{j=1}^m (\otimes_{i=1}^n f_{i,j})$ is an element of $\otimes_A^n K$ then an element σ in the group $\mathfrak{S}_n$ acts by $\sigma(\sum_{j=1}^m \otimes_{i=1}^n f_{i,j}) = \sum_{j=1}^m (\otimes_{i=1}^n f_{\sigma(i),j})$. The subring of symmetric tensors is written as $\otimes_A^{(n)} K$.

Proposition 6.4. *Let K be an A-algebra, and let $U \subseteq K$ be a multiplicatively closed set. For every positive integer n we define $U(n) \subseteq \otimes_A^{(n)} K$ as the multiplicatively closed set $U(n) = \{f \otimes \cdots \otimes f \mid f \in U \subseteq K\}$. Then we have a canonical isomorphism $\otimes_A^{(n)} K_U \cong \big(\otimes_A^{(n)} K\big)_{U(n)}$. In particular we have that*

$$(\otimes_A^{(n)} A[X])_{U(n)} \cong \otimes_A^{(n)} A[X]_U.$$

Proof. We define the multiplicatively closed set $U^n \subseteq \otimes_A^n K$ by

$$U^n = \{f_1 \otimes \cdots \otimes f_n \mid f_i \in U \subseteq K, \text{ for } i = 1, \ldots, n.\}$$

It follows by repeated use of Proposition (6.1) that we have a canonical isomorphism $\otimes_A^n K_U \cong (\otimes_A^n K)_{U^n}$. We have that $\otimes_A^{(n)} K_U$ is the ring of invariants of $\otimes_A^n K_U$

We will show that the ring of invariants of $(\otimes_A^n K)_{U^n}$ is $(\otimes_A^{(n)} K)_{U(n)}$. Let $U^{\mathfrak{S}} = U^n \cap (\otimes_A^{(n)} K)$.

Recall ([1], Exercise 12, p. 68) that if G is a finite group acting on a ring B, such that $G(U) \subseteq U$ for a multiplicatively closed set $U \subseteq B$. Then we have that $B^G_{U^G} \cong (B_U)^G$, where $U^G = U \cap B^G$

Hence we have that the ring of invariants of $(\otimes_A^n K)_{U^n}$ is $(\otimes_A^{(n)} K)_{U^{\mathfrak{S}}}$. Clearly we have $U(n) \subseteq U^{\mathfrak{S}}$. Our Proposition is proven if we show that the ring of invariants of $(\otimes_A^n K)_{U^n}$ is isomorphic to $(\otimes_A^{(n)} K)_{U^{\mathfrak{S}}}$. By Lemma (6.2) it suffices to show that for any element $f \in U^{\mathfrak{S}}$, there is an element H in $U^{\mathfrak{S}}$ such that the product is in $U(n)$. Let $f_1 \otimes \cdots \otimes f_n$ be an element of $U^{\mathfrak{S}} \subseteq U^n$. We have that

$$\prod_{i=1}^{n} f_i \otimes \cdots \otimes f_i = \big(f_1 \otimes \cdots \otimes f_n\big) H, \tag{6.4.1}$$

for some H in $\otimes_A^n K$. We have that H is in U^n, and we have that the product (6.4.1) is in $U(n)$. Since the product (6.4.1) is symmetric and the element $f_1 \otimes \cdots \otimes f_n$ is symmetric by assumption, it follows that H is in $U^{\mathfrak{S}}$. We have proven the Proposition.

§7. The addition map.

In Section (7) we show first that $\otimes_A^{(n)} A[X] \otimes_A A[X]$ is isomorphic to $V_{n+1,X}$. Thereafter we show that the isomorphism of $V^U_{n+1,X}$ and $(\otimes_A^{(n)} A[X])_{U(n)} \otimes_A A[X]_U$ follows by localization.

7.1. Definition. We have that $\otimes_A^{(n)} A[X]$ is the polynomial ring over A in the elementary symmetric functions $s_{1,n}(X), \dots, s_{n,n}(X)$ in the variables $X_1, \dots, X_n$. An A-algebra homomorphism from $\otimes_A^{(n)} A[X]$ to an A-algebra K is determined by its action on $s_{1,n}(X), \dots, s_{n,n}(X)$. For every positive integer n we define the *addition* map

$$a_n : \otimes_A^{(n)} A[X] \to \otimes_A^{(n-1)} A[X] \otimes_A A[X], \tag{7.1.1}$$

by sending $s_{i,n}(X)$ to $s_{i,n-1}(X) + s_{i-1,n-1}(X)X$ for every $i = 1, \dots, n$. As a convention we let $s_{0,n}(X) = 1$ and $s_{n,n-1}(X) = 0$ for all n, and we set $\otimes_A^{(0)} A[X] = A$. We denote with

$$\hat{a}_n : \otimes_A^{(n)} A[X] \otimes_A A[X] \to \otimes_A^{(n-1)} A[X] \otimes_A A[X]$$

the $A[X]$-algebra induced by the addition map.

Lemma 7.2. *For all positive integers n we define the $A[X]$-algebra homomorphism*

$$p_n : \otimes_A^{(n-1)} A[X] \otimes_A A[X] \to \otimes_A^{(n)} A[X] \otimes_A A[X]$$

recursively by $p_n(s_{i,n-1}(X)) = s_{i,n}(X) - p_n(s_{i-1,n-1}(X))X$ for $i = 1, \dots, n$ and where $p_n(s_{0,n-1}(X)) = p_n(1) = 1$. Then the following three assertions hold.

1. *We have that the composite map $p_n \circ \hat{a}_n$ is the identity map.*
2. *We have that $\hat{a}_n \circ p_n(s_{i,n-1}(X)) = s_{i,n-1}(X)$ for all $i = 1, \dots, n-1$.*
3. *The kernel of $\hat{a}_n$ is generated by*

Proof. We first prove Assertion (1). It is enough to show that $p_n \circ \hat{a}_n$ is the identity on the elementary symmetric functions $s_{1,n}(X), \dots, s_{n,n}(X)$. For each $i = 1, \dots, n$ we have that

$$\begin{aligned} p_n \circ \hat{a}_n(s_{i,n}(X)) &= p_n\big(s_{i,n-1}(X)\big) + p_n\big(s_{i-1,n-1}(X)X\big) \\ &= s_{i,n}(X) - p_n(s_{i-1,n-1}(X)X) + p_n(s_{i-1,n-1}(X)X) \\ &= s_{i,n}(X). \end{aligned}$$

We have proved the first Assertion. The second Assertion is proven by induction on i. For $i = 1$ we get by definition that $\hat{a}_n \circ p_n(s_{1,n-1}(X)) = \hat{a}_n(s_{1,n}(X) - X) = s_{1,n-1}(X)$. Assume as the induction hypothesis that $\hat{a}_n \circ p_n(s_{i,n-1}(X)) = s_{i,n-1}(X)$ for $i \geq 1$. We then get that

$$\begin{aligned} \hat{a}_n \circ p_n(s_{i+1,n-1}(X)) &= \hat{a}_n\big(s_{i+1,n}(X) - p_n(s_{i,n-1}(X)X)\big) \\ &= s_{i+1,n-1}(X) + s_{i,n-1}(X)X - \hat{a}_n \circ p_n(s_{i,n-1}(X)X) \\ &= s_{i+1,n-1}(X). \end{aligned}$$

Thus we have proven Assertion (2). To prove the last Assertion we first show that $\Delta_{n,X}(X)$ is in the kernel of $\hat{a}_n$. We have that

$$\begin{aligned} \hat{a}_n(\Delta_{n,X}(X)) &= X^n + \sum_{i=1}^{n} (-1)^i \hat{a}_n(s_{i,n}(X)X^{n-i}) \\ &= X^n + \sum_{i=1}^{n} (-1)^i \big(s_{i,n-1}(X) + s_{i-1,n-1}(X)X\big)X^{n-i} \end{aligned} \tag{7.2.1}$$

By definition we have that $s_{0,n-1}(X) = 1$ and that $s_{n,n-1}(X) = 0$. Thus it follows from (7.2.1) that $\Delta_{n,X}(X)$ is in the kernel of $\hat{a}_n$. By Assertion (1) we have that $\hat{a}_n \circ p_n$ is the identity. Since $\Delta_{n,X}(X)$ is in the kernel of $\hat{a}_n$ we get an induced homomorphism

$$\hat{p}_n : \big(\otimes_A^{(n-1)} A[X]\big) \otimes_A A[X] \to V_{n,X}, \tag{7.2.2}$$

where $V_{n,X} = \otimes_A^{(n)} A[X] \otimes_A A[X]/(\Delta_{n,X}(X))$. We have that $\hat{a}_n \circ \hat{p}_n$ is the identity map. In particular $\hat{a}_n$ is surjective. Our claim follows if we prove that the map $\hat{p}_n$ is surjective. The $A[X]$-algebra $V_{n,X}$ is generated by the classes of $s_{1,n}(X), \dots, s_{n,n}(X)$. It follows from Assertion (2) that we only need to show that the class of $s_{n,n}(X)$ in $V_{n,X}$ is in the image of $\hat{p}_n$. However we have that

$$X^n + s_{1,n}(X)X^{n-1} + \cdots + (-1)^{n-1}s_{n-1,n}(X)X = (-1)^{n-1}s_{n,n}(X)$$

in $V_{n,X}$. Hence we have that $\hat{p}_n$ is surjective. We have proven the Lemma.

Lemma 7.3. *Let n be a positive integer. For all integers $i = 1, \dots, n$ and for all $f(X)$ in $A[X]$ we have that that the homomorphism a_n sends the element $s_{i,n}(f(X))$ in $\otimes_A^{(n)} A[X]$ to $s_{i,n-1}(f(X)) + s_{i-1,n-1}(f(X))f(X)$ in $\otimes_A^{(n-1)XS} A[X] \otimes_A A[X]$. In particular we have that $f(X) \otimes \cdots \otimes f(X)$ in $\otimes_A^{(n)} A[X]$ is mapped to*

$$a_n\big(f(X) \otimes \cdots \otimes f(X)\big) = \big(f(X) \otimes \cdots \otimes f(X)\big) \otimes f(X)$$

.

Proof. We have a natural identification $\iota : \otimes_A^n A[X] \to \otimes_A^{n-1} A[X] \otimes_A A[X]$ which sends X_i to X_i when $i = 1, \dots, n-1$ and X_n to X. For every $i = 1, \dots, n$ we have that

$$\begin{aligned} s_{i,n}(X) &= \sum_{0<k_1<\cdots<k_i<n+1} X_{k_1} \cdots X_{k_i} \\ &= \sum_{0<k_1<\cdots<k_i<n} X_{k_1} \cdots X_{k_i} + \sum_{0<k_1<\cdots<k_{i-1}<n} X_{k_1} \cdots X_{k_{i-1}} X_n \\ &= s_{i,n-1}(X) + s_{i-1,n-1}(X) X_n. \end{aligned} \tag{7.3.1}$$

It follows from (7.3.1) that $\iota(s_{i,n}(f(X))) = s_{i,n-1}(f(X)) + s_{i-1,n-1}(f(X))f(X)$. From the definition of the addition map and (7.3.1) we have that the restriction of ι to the subring $\otimes_A^{(n)} A[X] \subseteq \otimes_A^n A[X]$ coincides with a_n, and our claim follows.

Recall that we defined (5.2.2) the ring $V_{n,X}^U$ as the localization of the $\otimes_A^{(n)} A[X]$-algebra $V_{n,X}$ with respect to the multiplicatively closed subset $U(n) \subseteq \otimes_A^{(n)} A[X]$. We saw in Proposition (6.4) that the fraction ring $(\otimes_A^{(n)} A[X])_{U(n)}$ was naturally identified with the ring of symmetric tensors $\otimes_A^{(n)} A[X]_U$. In the next Proposition we show a similar behavior for $V_{n,X}^U$.

Proposition 7.4. *Let $U \subseteq A[X]$ be a multiplicatively closed subset. We have that the addition map $a_n : \otimes_A^{(n)} A[X] \to \otimes_A^{(n-1)} A[X] \otimes_A A[X]$ induces an isomorphism*

$$V_{n,X}^U \cong \big(\otimes_A^{(n-1)} A[X]_U \big) \otimes_A A[X]_U.$$

Proof. The homomorphism $\hat{a}_n : \otimes_A^{(n)} A[X] \otimes_A A[X] \to \otimes_A^{(n-1)} A[X] \otimes_A A[X]$ is surjective by Assertion (1) of Lemma (7.2). By Assertion (3) of Lemma (7.2) the kernel of $\hat{a}_n$ is generated by $\Delta_{n,X}(X)$. Thus we have that the addition map induces an isomorphism

$$V_{n,X} \cong (\otimes_A^{(n-1)} A[X]) \otimes_A A[X]. \tag{7.4.1}$$

When we localize the $\otimes_A^{(n)} A[X]$-module $V_{n,X}$ with respect to the multiplicatively closed subset $U(n) \subseteq \otimes_A^{(n)} A[X]$ we get by definition $V_{n,X}^U$. Consequently the proof of the Proposition will be complete when we show that tensoring the right term in (7.4.1) with $\big(\otimes_A^{(n)} A[X]\big)_{U(n)}$ gives $(\otimes_A^{(n-1)} A[X]_U) \otimes_A A[X]_U$.

Let $W \subseteq B = \otimes_A^{(n-1)} A[X] \otimes_A A[X]$ be the multiplicatively closed set

$$W = \{(f(X) \otimes \cdots \otimes f(X)) \otimes f(X) \mid f(X) \in U \subseteq A[X]\}.$$

It follows from Lemma (7.3) that W is the image of the multiplicatively closed subset $U(n) \subseteq \otimes_A^{(n)} A[X]$ by the addition map $a_n : \otimes_A^{(n)} A[X] \to B$. Hence we have that $B \otimes_{\otimes_A^{(n)} A[X]} \big(\otimes_A^{(n)} A[X]\big)_{U(n)} = B_W$. Furthermore we have by Proposition (6.4) that $\otimes_A^{(n-1)} A[X]_U$ is the fraction ring of $\otimes_A^{(n-1)} A[X]$ with respect to the

multiplicatively closed subset $U(n-1)$. It then follows from Proposition (6.1) that we have

$$(\otimes_A^{(n-1)} A[X]_U) \otimes_A A[X]_U \cong (\otimes_A^{(n-1)} A[X] \otimes_A A[X])_{U(n-1)\cdot U},$$

where $U(n-1)\cdot U = \{F(X)\otimes g(X) \mid F(X) \in U(n-1), g(X) \in U\}$. It is clear that we have $W \subseteq U(n-1)\cdot U$. Hence to prove the Proposition it is by Lemma (6.2) sufficient to show that for any f in $U(n-1)\cdot U$ there is g in $U(n-1)\cdot U$ such that the product is contained in W.

Given $F(X)\otimes g(X)$ in $U(n-1)\cdot U$, where $F(X) = f(X)\otimes\cdots\otimes f(X)$ is in $U(n-1)$, and where $g(X)$ is in U. In the ring B we have the element $g(X)\otimes\cdots\otimes g(X)\otimes f(X)$, easily seen to be in $U(n-1)\cdot U$. We have that

$$\begin{aligned} \Big(F(X)\otimes g(X)\Big)\cdot\big(g(X)\otimes\cdots\otimes g(X)\otimes f(X)\big) \\ f(X)g(X)\otimes\cdots\otimes f(X)g(X), \end{aligned}$$

which is an element of W. We have proven the Proposition.

Remark. B. Iversen defines (See [9], page 3, Section (1.4)) for any flat A-module C a canonical map

$$\otimes_A^{(n+m)} C \to \big(\otimes_A^{(n)} C\big) \otimes_A \big(\otimes_A^{(m)} C\big),$$

which sends $(x_1\otimes\cdots\otimes x_{n+m})$ to $(x_1\otimes\cdots\otimes x_m)\otimes(x_{m+1}\otimes\cdots\otimes x_{n+m})$. When applied to our situation with $C = A[X]$, and $m = 1$, we have a map $\otimes_A^{(n+1)} A[X] \to \otimes_A^{(n)} A[X]\otimes_A A[X]$. It follows from Lemma (7.3) that when applied to our situation then our addition map a_{n+1} coincides with the canonical map of B. Iversen.

§8. - Application to Hilbert schemes.

8.1. The Hilbert functor of points. Let A be a commutative ring. For any A-algebra $A \to K$ we define the Hilbert functor $\mathcal{H}ilb^n_{K/A}$ of n-points on $\operatorname{Spec}(K)$ as the contravariant functor from the category of A-schemes to sets, mapping an A-scheme $T \to \operatorname{Spec}(A)$ to the set

$$\mathcal{H}ilb^n_{K/A}(T) = \left\{ \begin{array}{l} \text{Closed subschemes } Z \subseteq T \times_A \operatorname{Spec}(K) \text{ such that} \\ \text{the projection map } p : Z \to T \text{ is flat and where} \\ \text{the global sections of } p^{-1}(t) \text{ is of dimension } n \text{ as} \\ \text{a } \kappa(t)\text{-vector space, for all points } t \in T. \end{array} \right\}$$

We call $\mathcal{H}ilb^n_{K/A}$ the *Hilbert functor* of n-points on $\operatorname{Spec}(K)$. In those cases when the functor $\mathcal{H}ilb^n_{K/A}$ is representable we call the representing scheme the *Hilbert scheme* of n-points on $\operatorname{Spec}(K)$.

If K is an A-algebra, we have the A-algebra of symmetric tensors $\otimes_A^{(n)} K$. We define the A-scheme

$$\operatorname{Symm}^n_A(K) = \operatorname{Spec}(\otimes_A^{(n)} K).$$

Theorem 8.2. *Let $U \subseteq A[X]$ be a multiplicatively closed subset of the polynomial ring in the variable X over a ring A. Let n be a fixed positive integer. Then we have that the A-scheme $Symm_A^n(A[X]_U) \to Spec(A)$ represents the functor $\mathcal{H}ilb^n_{A[X]_U/A}$. The universal family is given as*

$$Symm_A^{n-1}(A[X]_U) \times_A Spec(A[X]_U) \to Symm_A^n(A[X]_U).$$

Proof. First we show that $\mathrm{Symm}_A^{n-1}(A[X]_U) \times_A \mathrm{Spec}(A[X]_U)$ is an element of $\mathcal{H}ilb^n_{A[X]_U/A}(\mathrm{Symm}_A^n(A[X]_U))$. By Proposition (7.4) we have that the coordinate ring of $\mathrm{Symm}_A^{n-1}(A[X]_U) \times_A \mathrm{Spec}(A[X]_U)$ is isomorphic to $V^U_{n,X}$. As a consequence of Theorem (5.3) we have that $V^U_{n,X}$ is a free, rank n residue of $(\otimes_A^{(n)} A[X])_{U(n)} \otimes_A A[X]_U$. Finally we have by Proposition (6.4) that $(\otimes_A^{(n)} A[X])_{U(n)} \cong \otimes_A^{(n)} A[X]_U$. Thus we have that $\mathrm{Symm}_A^{(n-1)}(A[X]_U) \times_A \mathrm{Spec}(A[X]_U)$ is a $\mathrm{Symm}_A^n(A[X]_U)$-valued point of the Hilbert functor of n-points on $A[X]_U$.

The next step in the proof is to show that the induced natural transformation of functors $\mathrm{Hom}(-, \mathrm{Symm}_A^n(A[X]_U)) \to \mathcal{H}ilb^n_{A[X]_U/A}$ is an isomorphism. Given a A-scheme T and let Z be a T-valued point of $\mathcal{H}ilb^n_{A[X]_U/A}$. Let $\mathrm{Spec}(K) \subseteq T$ be an open affine subscheme. Let $\varphi : A \to K$ be the A-algebra homomorphism corresponding to the structure map $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$. Let the inverse image $Z \times_T \mathrm{Spec}(K)$ be given by the ideal $I \subseteq K \otimes_A A[X]_U$.

We have that $K \otimes_A A[X]_U \cong K[X]_{U^\varphi}$, where $U^\varphi \subseteq K[X]$ is the image of the multiplicatively closed set $U \subseteq A[X]$ under the induced map $A[X] \to K[X]$. Hence the K-algebra $K \otimes_A A[X]_U$ is essentially of finite type. By assumption we have that the K-module $M = K \otimes_A A[X]_U/I$ is a flat K-module such that for each prime ideal P in K we have that the $\kappa(P) = K_P/PK_P$-vector space $M \otimes \kappa(P)$ is of dimension n. It follows (see [12], Theorem (3.5)) that M is locally a free K-module. By possibly shrinking the open set $\mathrm{Spec}(K) \subseteq T$, we may assume that M is a free K-module of rank n.

We have that there exist a covering $\{U_i\}_{i \in \mathcal{I}}$ of T, where $U_i \subseteq T$ is open and affine, such that for every $i \in \mathcal{I}$ we have that $p_{i*}(\mathcal{O}_{Z \times_T U_i})$ is a free $\mathcal{O}_{U_i}$-module of rank n. Here $p_i : U_i \times_T Z \to U_i$ is the projection map. It follows by Theorem (6.3) that there exist a unique A-morphism $f_i : U_i \to \mathrm{Symm}_A^n(A[X]_U)$ such that

$$U_i \times_T Z \cong U_i \times_{\mathrm{Symm}_A^n(A[X]_U)} \mathrm{Symm}_A^{n-1}(A[X]_U) \times_A \mathrm{Spec}(A[X]_U), \tag{8.2.1}$$

for every $i \in \mathcal{I}$. It follows from the uniqueness of the morphisms f_i that they glue together and give a unique morphism $f_Z : T \to \mathrm{Symm}_A^n(A[X]_U)$ such that the element $Z \in \mathcal{H}ilb^n_{A[X]_U/A}(T)$ is the pull-back of $\mathrm{Symm}_A^{(n-1)}(A[X]_U) \times_A \mathrm{Spec}(A[X]_U)$ by the morphism f_Z. We have proven the Theorem.

Remark. Given a projective morphism $C \to S$, where we fix an embedding of C in some projective N-space $\mathbf{P}^N_S$ over S. Consider now only S-schemes that are locally noetherian. From A. Grothendiecks general theory of Hilbert functors, we have that $\mathcal{H}ilb^n_{C/S}$ is representable (and in fact projective) [5]. In the special case when $C \to S$ is projective, smooth and of relative dimension 1 over S, it was remarked (see [5], p. 275) that the Hilbert functor $\mathcal{H}ilb^n_{C/S}$ is represented by $\mathrm{Symm}^n_S(C)$.

Remark. A comparison with B. Iversens theory of n-fold sections. Let $C \to S$ be a flat morphism of schemes. If $T \to S$ is a morphism of schemes then an n-fold section of C over T is a closed subscheme $Z \subseteq T \times_S C$ such that the projection map $Z \to T$ is finite, flat and of rank n. We denote with $\mathcal{F}^n_{C/S}(T)$ the set of n-fold sections of C over T. It is clear that we have a contravariant functor $\mathcal{F}^n_{C/S}$ from the category of S-schemes to sets, [9].

We will compare the functor $\mathcal{F}^n_{C/S}$ with the Hilbert functor $\mathcal{H}ilb^n_{K/A}$ in the specific situation when $C = \operatorname{Spec}(K)$ and $S = \operatorname{Spec}(A)$. It is clear that $\mathcal{F}^n_{\operatorname{Spec}(K)/\operatorname{Spec}(A)}$ is a subfunctor of $\mathcal{H}ilb^n_{K/A}$.

I will show, by repeating an argument given in the proof of the Theorem, that when K is a flat A-algebra, essentially of finite type, then the two functors $\mathcal{F}^n_{\operatorname{Spec}(K)/\operatorname{Spec}(A)}$ and $\mathcal{H}ilb^n_{K/A}$ are naturally identified. Note that if Z is a T-valued point of $\mathcal{H}ilb^n_{K/A}$ then it is not obvious that the projection map $Z \to T$ is finite.

Let $\operatorname{Spec}(R)$ be an open subscheme of an A-scheme T. Let Z be an T-valued point of $\mathcal{H}ilb^n_{K/A}$. We have that Z is a closed subscheme of $T \times_A \operatorname{Spec}(K)$, hence $Z_R = Z \times_T \operatorname{Spec}(R)$ is a closed subscheme of $\operatorname{Spec}(R) \times_A \operatorname{Spec}(K)$. Let $Z_R = \operatorname{Spec}(M)$. It follows from the definition of the Hilbert functor $\mathcal{H}ilb^n_{K/A}$ that M is a flat R-module such that for all prime ideal P in R we have that the $\kappa(P) = R_P/PR_P$-vector space $M \otimes_R \kappa(P)$ is of dimension n. Since K is essentially of finite type over A, it follows that M is essentially of finite type over R. Hence it follows ([12], Theorem (3.5)), that M is locally free over R. The rank of M as an R-module is clearly n, and consequently the T-valued point Z of $\mathcal{H}ilb^n_{K/A}$ is a T-valued point of $\mathcal{F}^n_{\operatorname{Spec}(K)/\operatorname{Spec}(A)}$.

We thus have two functors $\mathcal{F}^n_{C/S}$ and $\mathcal{H}ilb^n_{K/A}$ that are equal when K is a flat A-algebra, essentially of finite type. In particular the functors are equal when $K = A[X]_U$, a fraction ring of the polynomial ring in the variable X over A. It is therefore natural to compare the results of [9] with the ones have in the present paper. In particular there are two results that are closely related to the Main Theorem of the present paper.

One of the results which B. Iversen shows ([9], Theorem (3.4), p. 26) is the following.

Let $C \to S$ be a flat and finite morphism of schemes such that for all points $s \in S$, any finite set of points of the fiber over s is contained in an open affine subset of C, whose image by $C \to S$ is contained in an open affine of S ([9], page 21, Section (1.1)). If the canonical map $\operatorname{Symm}^{(n-1)}_S(C) \times_S C \to \operatorname{Symm}^{(n)}_S(C)$ is finite, flat and of rank n, then the functor $\mathcal{F}^n_{C/S}$ is represented by $\operatorname{Symm}^{(n)}_S(C)$ and where the universal family is $\operatorname{Symm}^{(n-1)}_S(C) \times_S C$. The canonical map is the one we discussed in the Remark of Section (7) in the present paper.

Hence, modulo the result (Theorem (3.4)) of B. Iversen, we could have shortened our proof of Theorem (8.2). Because to prove the statement of Theorem (8.2) it would be sufficient to show that $\operatorname{Symm}^n_A(A[X]_U) \times_A \operatorname{Spec}(A[X]_U)$ is an $\operatorname{Symm}^n_A(A[X]_U)$-valued point of $\mathcal{H}ilb^n_{A[X]_U/A}$.

The other result of [9] (Proposition (4.1), p. 29), which I want to mention is the case when $C \to S$ is a flat family of smooth curves (and where S is locally noetherian). With these extra assumptions B. Iversen shows that the canonical map $\operatorname{Symm}^{(n-1)}_S(C) \times_S C \to \operatorname{Symm}^{(n)}_S(C)$ is finite, flat and of rank n. In other

words the functor $\mathcal{F}^n_{C/S}$ is represented by $\mathrm{Symm}^{(n)}_S(C)$.

Comparing with our situation we have that the homomorphism $\mathrm{Spec}(A[X]_U) \to \mathrm{Spec}(A)$ is smooth, but the fibers are not necessarily of finite type, hence neither curves. When $S = \mathrm{Spec}(A)$ is noetherian and $A[X]_U$ is finitely generated as an A-algebra, or equivalently that $\mathrm{Spec}(A[X]_U)$ is an basic open subset of $\mathrm{Spec}(A[X])$, then our Theorem (8.2) is a consequence of [9] (Proposition (4.1), p. 29).

8.3. Example I: The Hilbert scheme of points on the affine line. When $U = \{1\} \subseteq A[X]$ is the trivial subset we have that $\mathrm{Spec}(A[X]_U) = \mathbf{A}^1_A$ is the affine line over A. The Hilbert scheme of n-points on $\mathbf{A}^1_A$ is given as $\mathrm{Spec}(\otimes^{(n)}_A A[X])$. The ring of symmetric functions $\otimes^{(n)}_A A[X]$ is the polynomial ring in n-variables over A. Thus the parameterizing scheme $\mathrm{Symm}^n_A(A[X])$ is simply the affine n-space $\mathbf{A}^n_A$ over A. Note that the only assumptions on the base ring A is that A is commutative and unitary.

8.4. Example II: The Hilbert scheme of points on open subsets of the line. Let the multiplicatively closed subset U be given by multiples of one element f in $A[X]$, that is $U = \{f^m\}_{m\geq 0}$. Then $\mathrm{Spec}(A[X]_U)$ is an basic open (possibly empty) subscheme of $\mathbf{A}^1_A$, the affine line over A.

The Hilbert scheme of n-points on $\mathrm{Spec}(A[X]_U)$ is given as the spectrum of $(\otimes^{(n)}_A A[X])_{U(n)}$, where $U(n) = \{(f \otimes \cdots \otimes f)^m\}_{m\geq 0}$. Hence we have that the Hilbert functor of n-points on an basic open subscheme of the line is represented by an open subscheme of the Hilbert scheme of n-points on the line.

8.5. Example III: The Hilbert scheme parameterizing finite length subschemes of the line with support at the origin. Let the base ring $A = k$ be a field, and let $U \subseteq k[X]$ be the set of polynomials $f(X)$ such that $f(0) \neq 0$. Thus $k[X]_U = k[X]_{(X)}$ is the local ring of the origin on the line, and the functor $\mathcal{H}ilb^n_{k[X]_{(X)}/k}$ parameterizes the length n subschemes of $\mathrm{Spec}(k[X]_{(X)})$. There is only one closed subscheme of $\mathrm{Spec}(k[X]_{(X)})$ of length n, namely the scheme given by the ideal $(X^n) \subseteq k[X]_{(X)}$. A situation which was studied in detail in [13].

8.6. Example IV: A Hilbert scheme without rational points. Assume that the base ring A is an integral domain. Let $U \subseteq A[X]$ be the set of non-zero polynomials. We have that $A[X]_U = A(X)$, the field of fractions of $A[X]$. We have that $\mathrm{Symm}^n_A(A(X))$ represents the Hilbert functor of n-points on $\mathrm{Spec}(A(X))$. Note that $\mathrm{Spec}(A(X))$ is just a point and that there exist no non-trivial subschemes of $\mathrm{Spec}(A(X))$. Consequently the Hilbert scheme of n-points on $\mathrm{Spec}(A(X))$ has no A-valued points. In particular if $A = k$ is a field, we have that the Hilbert scheme of n-points on $\mathrm{Spec}(k(X))$ has no k-rational points.

We will show that the parameterizing scheme $\mathrm{Symm}^n_k(k(X))$ is of dimension $(n-1)$ when the base ring $A = k$ is a field.

We have that the coordinate ring is, Proposition (6.4), $(\otimes^{(n)}_k k[X])_{U(n)}$, where $U(n)$ is the set of products of the form $f(X)\otimes\cdots\otimes f(X)$, with non-zero polynomials $f(X) \in k[X]$.

The ring of symmetric functions $\otimes^{(n)}_k k[X]$ is the polynomial ring in the variables $s_{1,n}(X), \ldots, s_{n,n}(X)$, and is consequently of dimension n. It is clear that to any maximal ideal P in $\otimes^{(n)}_k k[X]$, we can find an element of the form $f(X)\otimes\cdots\otimes f(X)$, with $f(X) \neq 0$, in P. Thus the extension of the maximal ideals in $\otimes^{(n)}_k k[X]$, in

the fraction ring $(\otimes_k^{(n)} k[X])_{U(n)}$ becomes the whole ring. A phenomena which we expected since the parameterizing scheme $\mathrm{Symm}_k^n(k(X))$ has no k-rational points.

We have that all the maximal ideals of $\otimes_k^{(n)} k[X]$ meets the set $U(n)$. It follows that we the dimension of $(\otimes_k^{(n)} k[X])_{U(n)}$ is at most $n-1$. Next we note that if $G(X_1, \dots, X_n)$ is an irreducible polynomial in the variables $X_1, \dots, X_n$, which is symmetric, then $G(X_1, \dots, X_n)$ correspond to a height 1 prime ideal in $\otimes_k^{(n)} k[X]$ which does not meet $U(n)$. If $n > 1$ clearly such functions exist. The elementary symmetric functions $s_{1,n}(X), \dots, s_{n-1,n}(X)$ are examples of such. Thus we have that the ideal P generated by $s_{1,n}(X), \dots, s_{n-1,n}(X)$ does not meet $U(n)$. Consequently the extension of P in the fraction ring $(\otimes_k^{(n)} k[X])_{U(n)}$ correspond to a prime ideal. We have that the localization of $\otimes_k^{(n)} k[X]$ in P is a local ring of dimension $(n-1)$. Since $(n-1)$ was an upper bound for the dimension of the fraction ring $(\otimes_k^{(n)} k[X])_{U(n)}$, it follows that $(n-1)$ is the dimension of $\mathrm{Symm}_k^n(k(X))$.

It may be surprising that we need a $(n-1)$ dimensional scheme to parameterize the empty set of closed subschemes of $\mathrm{Spec}(k(X))$ of finite length.

8.7. Example V: Hilbert schemes of one point. Let the fixed integer $n = 1$. For any multiplicatively closed subset $U \subseteq A[X]$ we have that $\mathrm{Symm}_A^1(A[X]_U) = \mathrm{Spec}(A[X]_U)$. Hence the scheme $\mathrm{Spec}(A[X]_U)$ itself is the Hilbert scheme of one point on $\mathrm{Spec}(A[X]_U)$. See also [10] (Corollary (2.3) of Proposition (2.2), p. 109) where S.L. Kleiman proves that for any S-scheme X the functor $\mathcal{H}ilb_{X/S}^1$ is represented by the scheme X, and where the universal family is given by the diagonal in $X \times_S X$.

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